

The null space of a linear map

1. The span of a collection of vectors is a subspace of the vector space in which they are defined. A basis for that span can be found by creating the corresponding matrix, performing Gaussian elimination until a matrix that is row-echelon form is found, and then choosing those vectors that correspond to columns with leading non-zero entries.

Given the following five vectors, what are all solutions to finding a linear combination of these vectors such that that linear combination equals the zero vector?

$$\begin{pmatrix} -0.8 \\ 0.2 \\ 2.0 \\ 0.6 \end{pmatrix}, \begin{pmatrix} -1.2 \\ 0.3 \\ 3.0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} -1.4 \\ 3.2 \\ 2.0 \\ 0.9 \end{pmatrix}, \begin{pmatrix} 2.2 \\ 1.4 \\ 4.0 \\ -0.7 \end{pmatrix}, \begin{pmatrix} 4.2 \\ 2.1 \\ 1.0 \\ -2.0 \end{pmatrix} \text{ and } \begin{pmatrix} -0.8 & -1.2 & -1.4 & 2.2 & 4.2 \\ 0.2 & 0.3 & 3.2 & 1.4 & 2.1 \\ 2.0 & 3.0 & 2.0 & 4.0 & 1.0 \\ 0.6 & 0.9 & 0.9 & -0.7 & -2.0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 2 & 4 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. What is a basis for the null space and the dimension of the null space for the matrix $A: \mathbf{R}^5 \rightarrow \mathbf{R}^4$ where

$$A = \begin{pmatrix} -0.8 & -1.2 & -1.4 & 2.2 & 4.2 \\ 0.2 & 0.3 & 3.2 & 1.4 & 2.1 \\ 2.0 & 3.0 & 2.0 & 4.0 & 1.0 \\ 0.6 & 0.9 & 0.9 & -0.7 & -2.0 \end{pmatrix}?$$

3. Is there just the trivial solution to finding a linear combination of the five vectors in Question 1 that equals the zero vector $\mathbf{0}_4$?

4. Is the null space of the linear map A in Question 2 just $\{\mathbf{0}_5\}$? That is, is A one-to-one?

5. For each of the following functions that map $\mathbf{R}$ to $\mathbf{R}$, choose those that are one-to-one:

$$e^x, x^2, x^3, x^3 + x, x^3 - x, \sin(x)$$

6. If $B: \mathbf{C}^3 \rightarrow \mathbf{C}^4$, answer the following:

- Suppose solving $B\mathbf{u} = \mathbf{0}_4$ has one free variable, is B one-to-one and what is the dimension of the null space?
- Suppose solving $B\mathbf{u} = \mathbf{0}_4$ has two free variables, is B one-to-one and what is the dimension of the range?
- Can B have no free variables and if so, what is the dimension of the null space?
- Can B have four free variables, and if so, what is the dimension of the null space?